a Local Search Heuristic for Facility Location Problems

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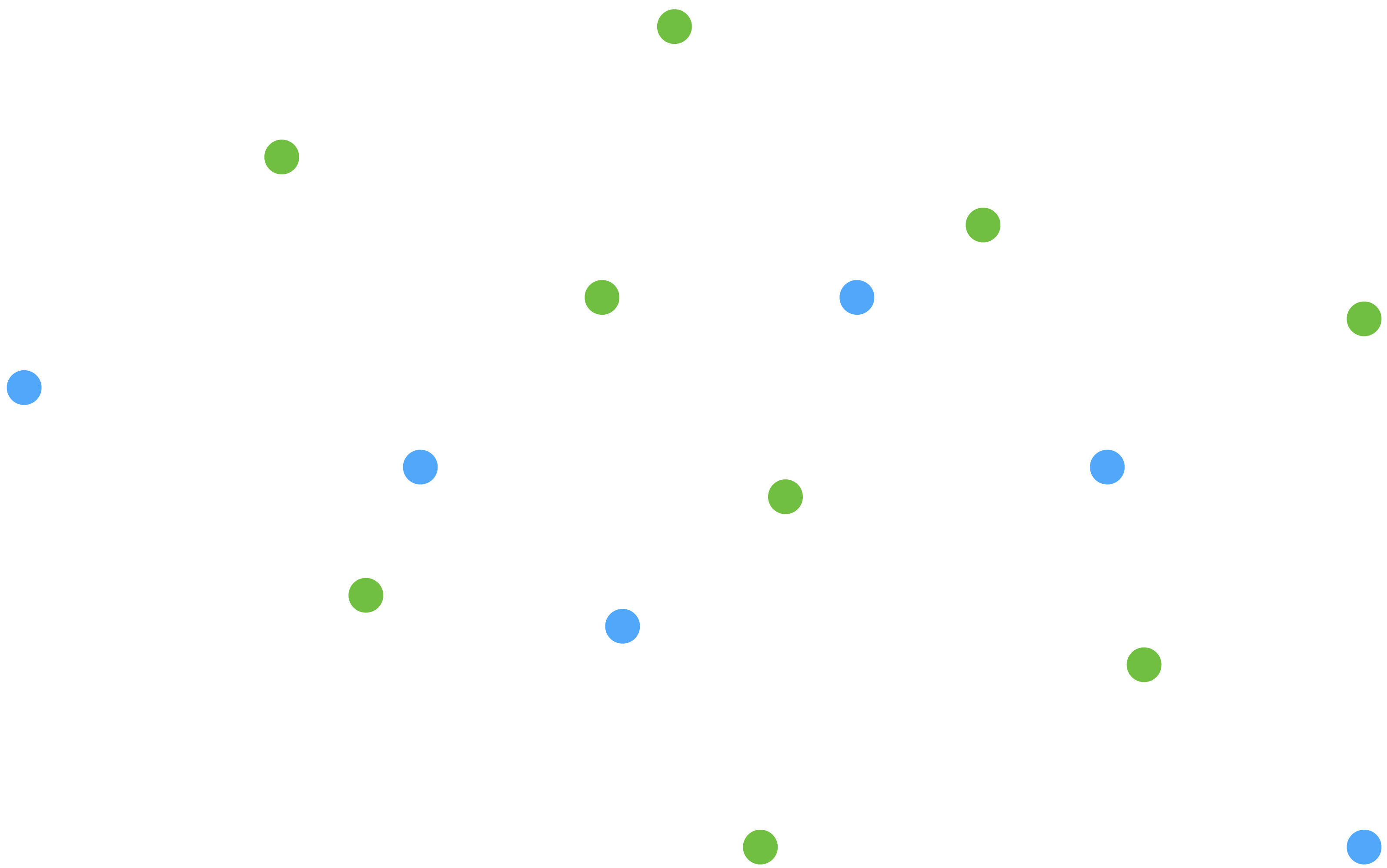
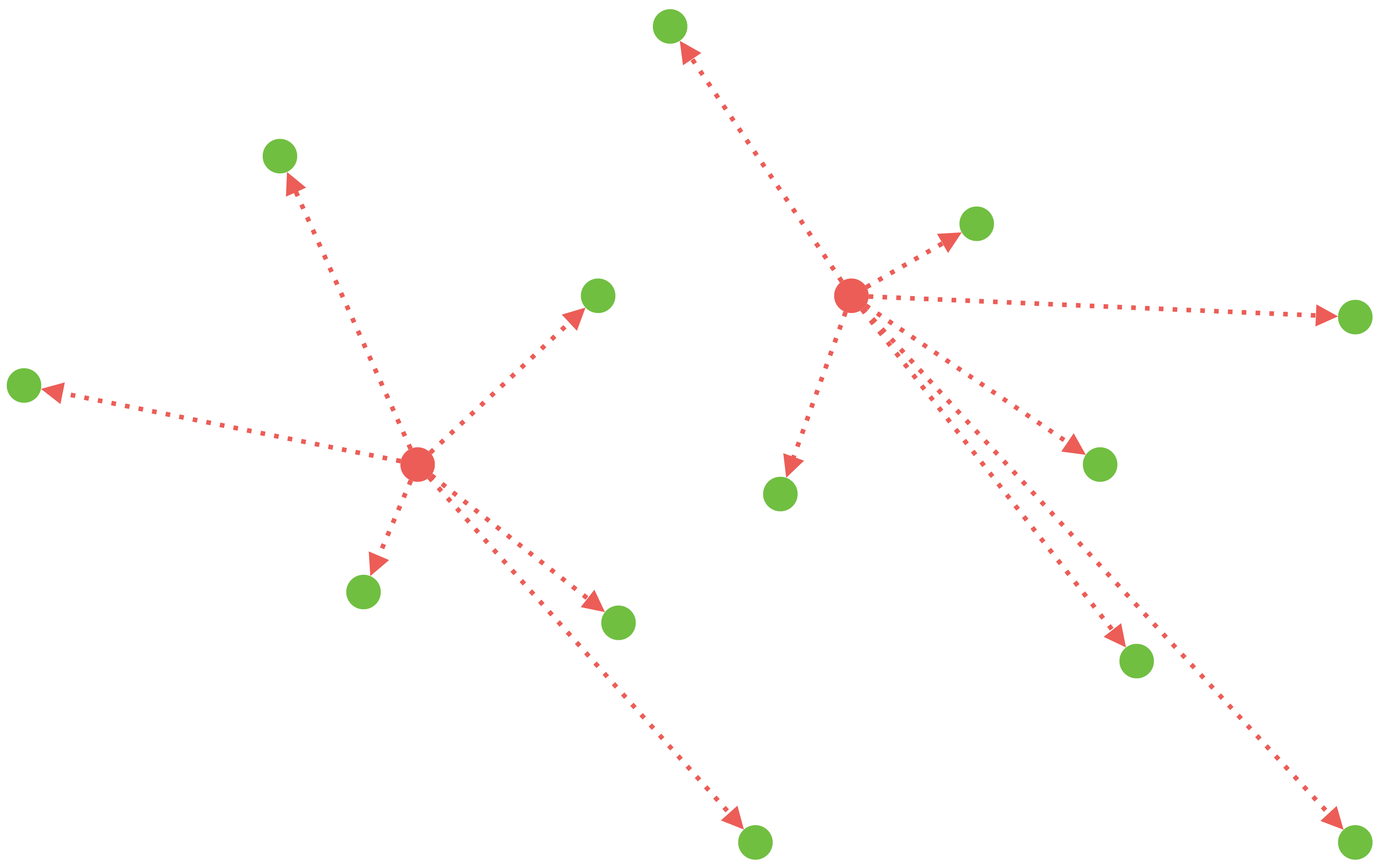
Introduction



Introduction



Outline

- Formal *definitions* of the facility location problems
- The local search *paradigm* (an iterative algorithm)
- Formal *proofs* of the facility location problems
  - Uncapcitated k-median problem
  - Capacitated k-median problem with splittable demands
- Approximation *ratio*

Background

$N = \{1, 2, \cdots, n\}$ is a set of locations

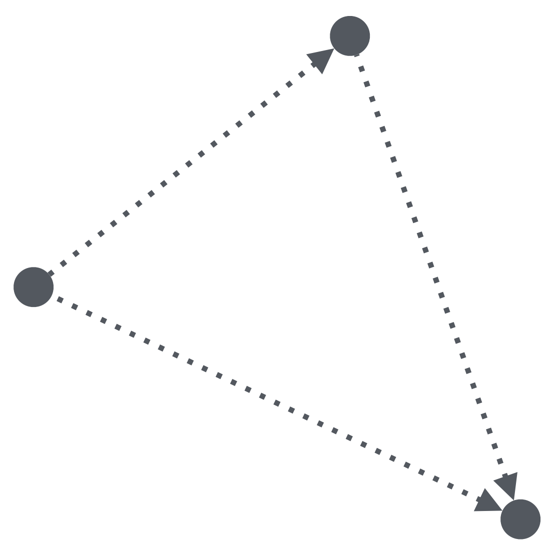
$F \subseteq N$ is a set of locations at which we may open a facility

Each location j has a demand d_j that must be shipped to j

c_{ij} is the cost of shipping a unit of demand from i to j

metric version

- cost is *nonnegative*
- cost is *symmetric*
- cost satisfies the *triangle inequality*



Uncapacitated k-median Problem

seek a set S of k open facilities

and

an assignment σ of locations to open facilities

to

minimize the shipping cost

$$C(S) = \min_{\sigma} C(S, \sigma)$$

$$C(S, \sigma) = \sum_{j \in N} d_j c_{j\sigma(j)}$$

assign each location to the closest open facility in S

Capacitated k-median Problem with Splittable Demands

a bound M on the capacity of any facility

seek a set S of k open facilities

an assignment σ of locations to open facilities

$$\sigma : N \times S \mapsto \mathbb{R}$$

Capacitated k-median Problem with Splittable Demands

a bound M on the capacity of any facility

seek a set S of k open facilities

an assignment σ of locations to open facilities

total demand shipped from any facility $\leq M$

$$\sum_j \sigma(j,i) \leq M, \quad \forall i$$

minimize the shipping cost

Capacitated k-median Problem with Splittable Demands

seek a set S of k open facilities

total demand shipped from any facility $\leq M$

minimize the shipping cost

$$C(S) = \min_{\sigma} C(S, \sigma)$$

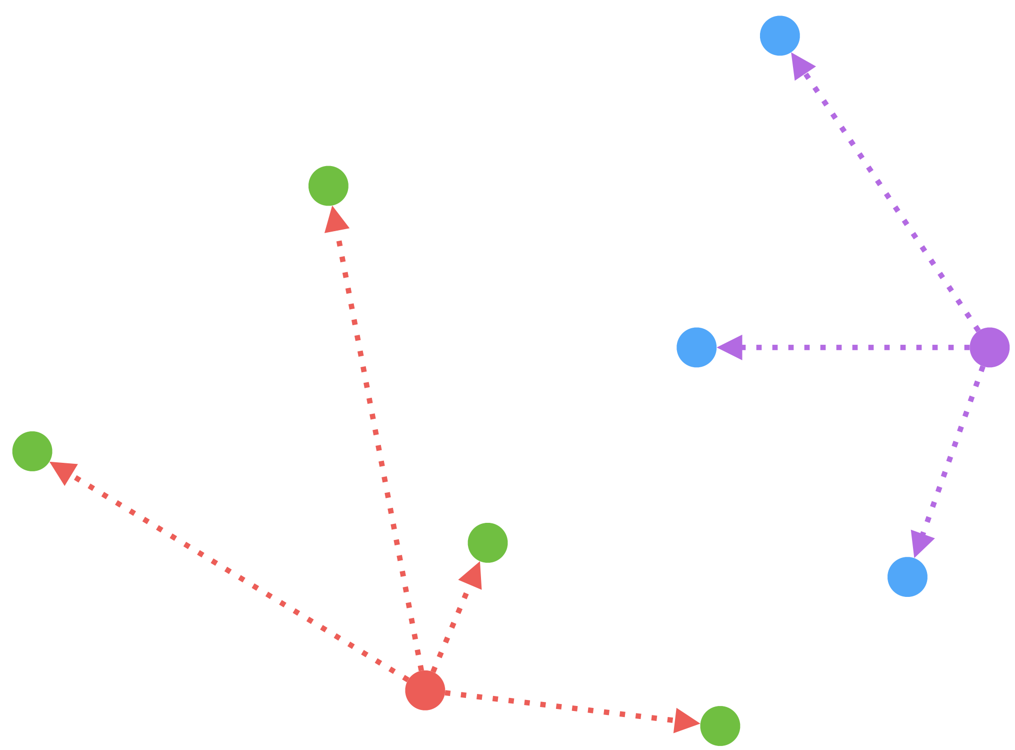
$$C(S, \sigma) = \sum_{i \in S} \sum_{j \in N} \sigma(j, i) c_{ij}$$

the corresponding optimal (splittable) assignment can be computed *in polynomial time* by reduction to the *transportation problem*

Definition

Given a set S of open facilities and an assignment σ

$D_i(S, \sigma)$ is the total demand shipped from facility $i \in S$



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total demand of ● is ● + ● + ● + ●

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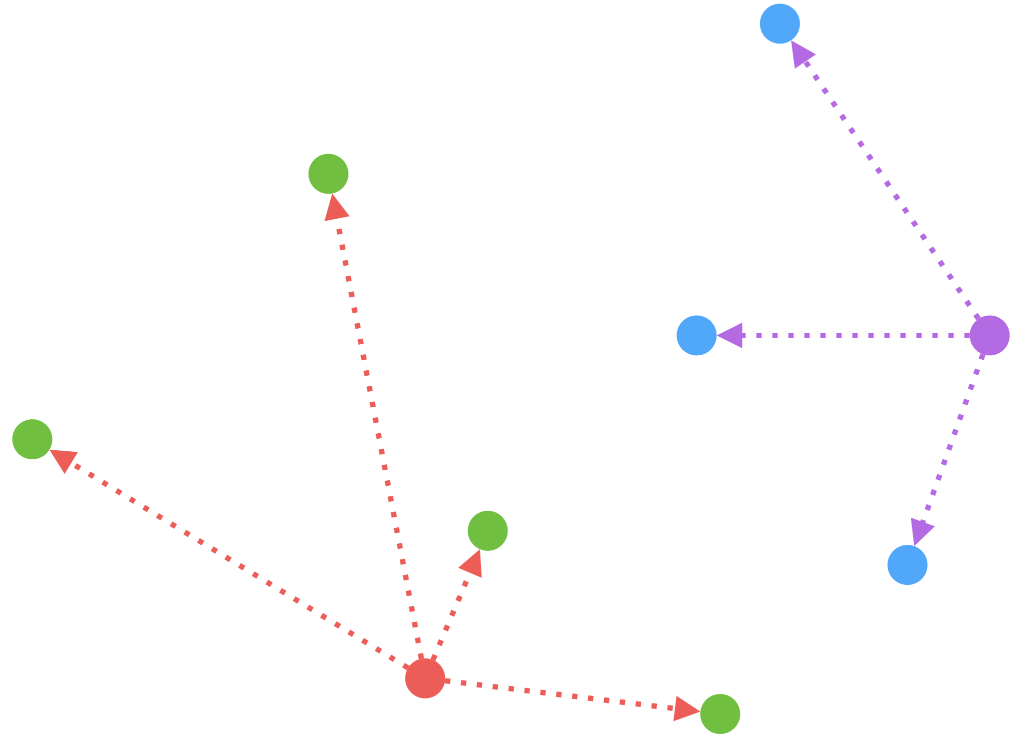
Definition

Given a set S of open facilities and an assignment σ

$N_i(S, \sigma)$ is the set of locations assigned to facility $i \in S$



Definition

Given a set S of open facilities and an assignment σ

$N_i(S, \sigma)$ is the set of locations assigned to facility $i \in S$

neighbor of ● is ● + ● + ● + ●

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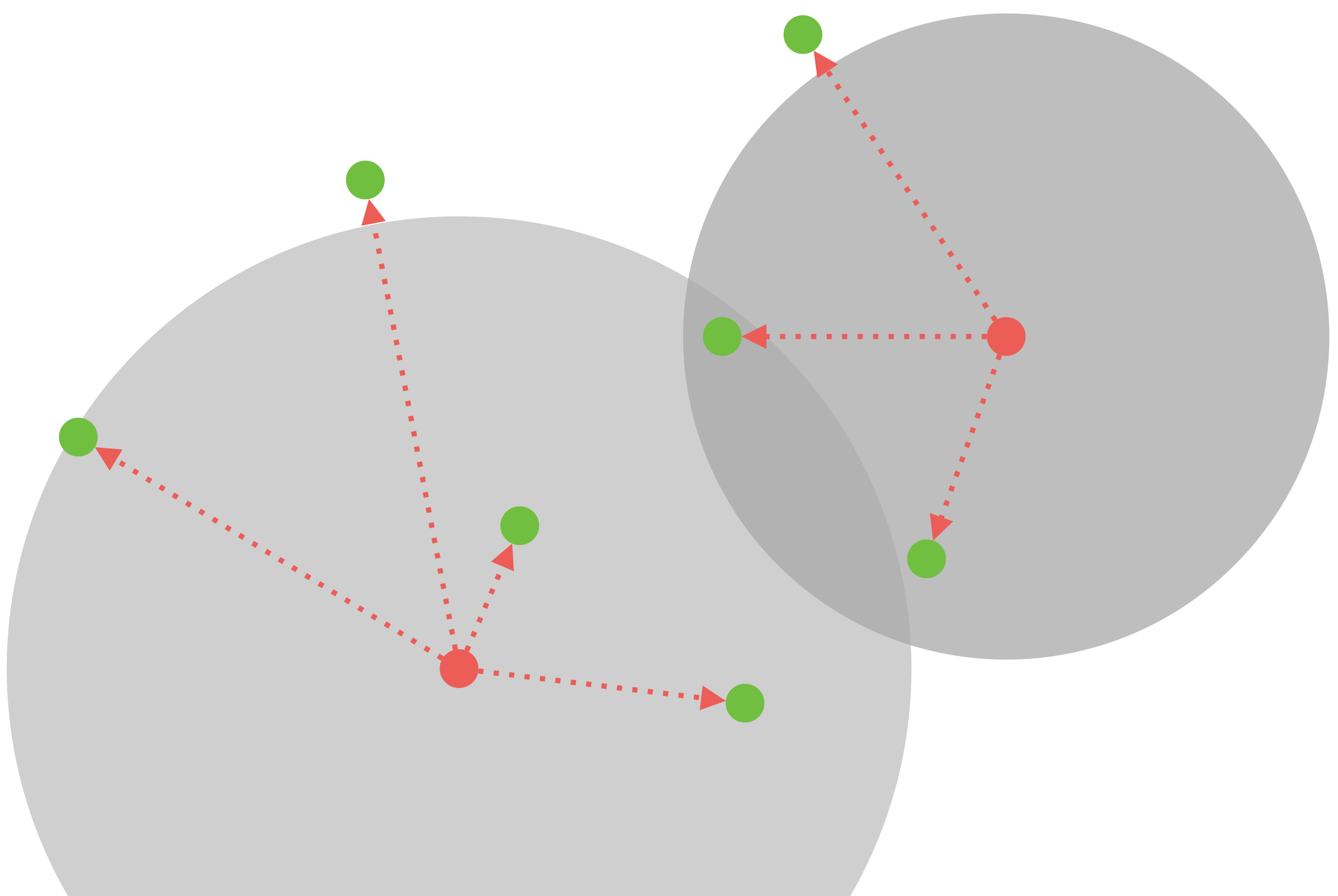
Definition

Given a set S of open facilities and an assignment σ

$B(i, r)$ is the set of locations j such that c_{ij} is at most r



Local Search Paradigm

```
Solve()
  pick a feasible solution
  while the computer works
    move to the best neighboring feasible solution
```

Technical Details

How to find an initial feasible solution?

Solve()
pick a *feasible* solution
while *the computer works*
move to the best **neighboring** *feasible* solution

Technical Details

How to find an initial feasible solution?

Select an arbitrary subset S of F

require that $|S| = (1+\alpha)k$

Solve()
pick a *feasible* solution while *the computer works*   move to the best **neighboring** *feasible* solution

Technical Details

What is a neighboring feasible solution?

Solve()

pick a *feasible* solution
while *the computer works*
move to the best **neighboring** *feasible* solution

Technical Details

What is a neighboring feasible solution?

The *neighborhood* of a feasible solution S is

$$\{T \subseteq F : |S \setminus T| = |T \setminus S| = 1\}$$

change a location of a facility

Solve()
pick a *feasible* solution while *the computer works* move to the best **neighboring** *feasible* solution

Solve()
pick a *feasible* solution while *the computer works*   move to the best **neighboring** *feasible* solution

Given any solution S such that $C(S) > (1+\epsilon)C(S^*)$

(require that $|S| = (1+\alpha)k$ for the k-median problems)

Target: there exists a solution T in the neighborhood of S

such that $C(T) \leq C(S)\left(1 - \frac{1}{p(n)}\right)$

Solve()
pick a *feasible* solution while *the computer works* move to the best **neighboring** *feasible* solution

Target: there exists a solution T in the neighborhood of S

such that $C(T) \leq C(S)\left(1 - \frac{1}{p(n)}\right)$

polynomial in the input size

Perform the local search $p(n) \log \frac{C(S)}{C(S^*)}$ times

#neighbors $= O(n^2)$

$\Rightarrow$ a solution that has cost at most $(1+\epsilon)C(S^*)$

$(1+\epsilon)$-approximation

Uncapacitated k-median Problem

Target: there exists a solution T in the neighborhood of S

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Theorem (*swapping facilities*)

Let S be any subset of F such that $|S| = (1+\alpha)k$ and $C(S) > (1+\beta)C(S^*)$. Then there exist $u \in S$ and $v \in F$ such that

$$C(S) - C(S+v-u) \geq \frac{C(S)}{p(n)}$$

Uncapacitated k-median Problem

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Adding a facility can get significant improvement
Dropping a facility without incurring a large increase

Lemma (*adding a facility*)
Let S be any subset of F such that $C(S) > (1+\beta)C(S^*)$.
Then there exist $v \in F$ such that

$$C(S) - C(S+v) \geq \frac{\beta C(S)}{(1+\beta)k}$$

$$\begin{aligned} C(S) - C(S^*) &= \sum_{j \in N} d_j (c_{j\sigma(j)} - c_{j\sigma^*(j)}) \\ &= \sum_{i \in S^*} \sum_{j \in N_i(S^*)} d_j (c_{j\sigma(j)} - c_{ji}) \end{aligned}$$

there exists $v \in S^*$ such that

$$\sum_{j \in N_v(S^*)} d_j (c_{j\sigma(j)} - c_{jv}) \geq \frac{C(S) - C(S^*)}{k}$$

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$$= C(S) - C(S+v, \sigma') \leq C(S) - C(S+v)$$

$$\sigma'(j) = \begin{cases} v & \text{if } j \in N_v(S^*) \\ \sigma(j) & \text{otherwise} \end{cases}$$

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$$\geq \frac{\beta C(S)}{(1+\beta)k} =$$

Lemma (*dropping a facilities*)
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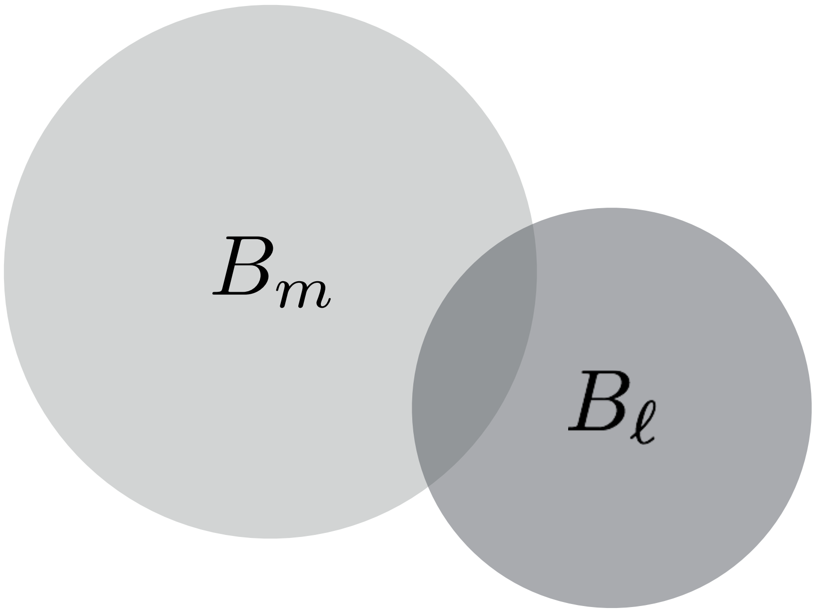
$$C(S-u) - C(S) \leq \frac{\beta C(S)}{(1+\beta)k} - \frac{C(S)}{p(n)}$$

$$L = \qquad r_i = \frac{L}{2D_i(S)} \qquad B_i = B(i, r_i)$$

Suppose B_m and B_ℓ overlap

$$C(S-m) - C(S) \leq C(S-m, \sigma') - C(S)$$

$$\sigma'(j) = \begin{cases} \ell & \text{if } \sigma(j) = m \\ \sigma(j) & \text{otherwise} \end{cases}$$



$r_m \geq r_\ell$

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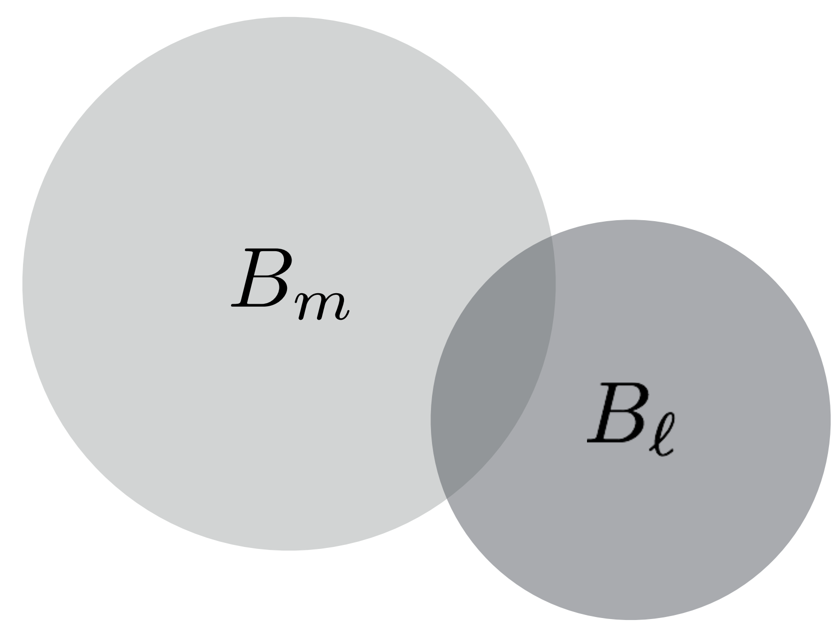
$$\begin{aligned} C(S-m) - C(S) &\leq C(S-m, \sigma') - C(S) \\ &\leq \sum_{j \in N_m(S)} d_j (c_{\ell j} - c_{mj}) \\ &\leq \sum_{j \in N_m(S)} 2 d_j r_m \end{aligned}$$



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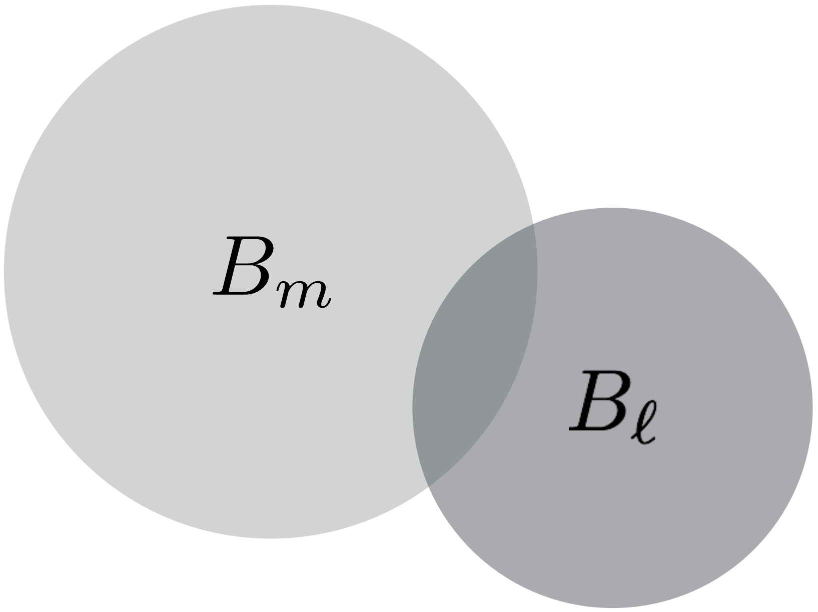
Suppose B_m and B_ℓ overlap

$$C(S-m) - C(S) \leq \sum_{j \in N_m(S)} 2d_j r_m$$

$$= 2D_m(S) r_m$$

$$= L$$



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$$Q_i(S) = \sum_{j \in N_i(S) \cap B(i, \mu r_i)} d_j$$

Suppose none of the B_i's overlap

$$S' = \left\{ i \in S : Q_i(S)(1-\mu)r_i \geq \frac{C(S)}{(1+\beta)\gamma k} \right\}$$

size of [highlighted box] $\geq (1+\gamma)k$

$$S'' = \{i \in S' : B(i, r_i) \cap S^* = \emptyset\}$$

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size of [blue box] $\geq (1+\gamma)k$

size of [red box] $\geq \gamma k$

size of [green box] $= k$

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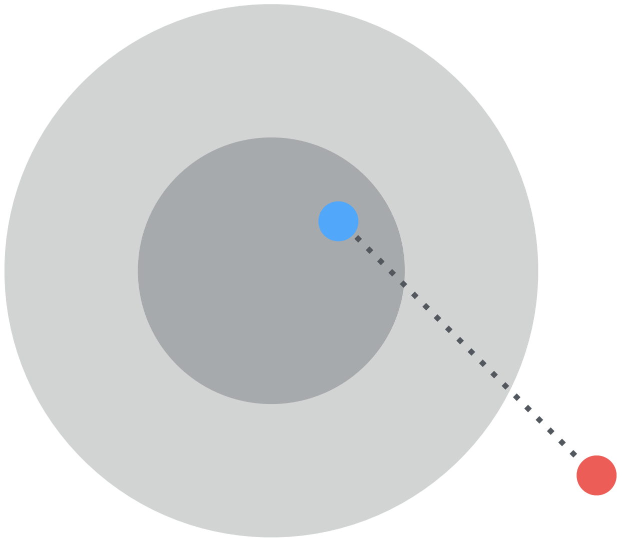
Consider the demands lie inside $B(i, \mu r_i)$ for $i \in S''$

size of [red box] $\geq \gamma k$

$$C(S^*) \geq \sum_{i \in S''} \sum_{j \in B(i,\mu r_i)} d_j (1-\mu) r_i$$

$$\geq \sum_{i \in S''} Q_i(S)(1-\mu) r_i$$



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size of [ ] $\geq \gamma k$

$$C(S^*) \geq \sum_{i \in S''} Q_i(S)(1-\mu)r_i \geq \sum_{i \in S''} \frac{C(S)}{(1+\beta)\gamma k}$$

$$= |S''| \frac{C(S)}{(1+\beta)\gamma k} \geq \frac{C(S)}{1+\beta}$$

Contradiction!

Uncapacitated k-median Problem

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The proof of the this **Lemma** (*adding a facility*)
is similar to that of **Lemma** (*adding a facility*)
mentioned in uncapacitated version

Lemma (*dropping a facility*)
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A facility $i \in S$ is light (under the corresponding optimal assignment σ) if the total demand shipped from i is at most $M/2$. Otherwise, it is heavy.

Suppose B_m and B_ℓ are light and overlap

Suppose none of the light B_i's overlap

Approximation Ratio

(a, b)-approximation algorithm

a *polynomial-time* algorithm using at most bk facilities

whose cost is at most $aC(S^*)$

For uncapacitated k-median

$$a = 1 + \epsilon \quad \Rightarrow \quad b = 1 + \left(9 + \frac{17}{\epsilon}\right)$$

$$a = 1 + \Theta\left(\frac{1}{\epsilon^3}\right) \quad \Leftarrow \quad b = 3 + \epsilon$$

Approximation Ratio

(a, b)-approximation algorithm

a *polynomial-time* algorithm using at most bk facilities

whose cost is at most $aC(S^*)$

For capacitated k-median

$$a = 1 + \epsilon \quad \Rightarrow \quad b = 1 + \left(11 + \frac{17}{\epsilon}\right)$$

$$a = 1 + \Theta\left(\frac{1}{\epsilon^3}\right) \quad \Leftarrow \quad b = 5 + \epsilon$$

Review

- Formal *definitions* of the facility location problems
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    - Capacitated k-median problem with splittable demands
- Approximation *ratio*

Q. & A.